CHAPTER 2
Time Value of Money

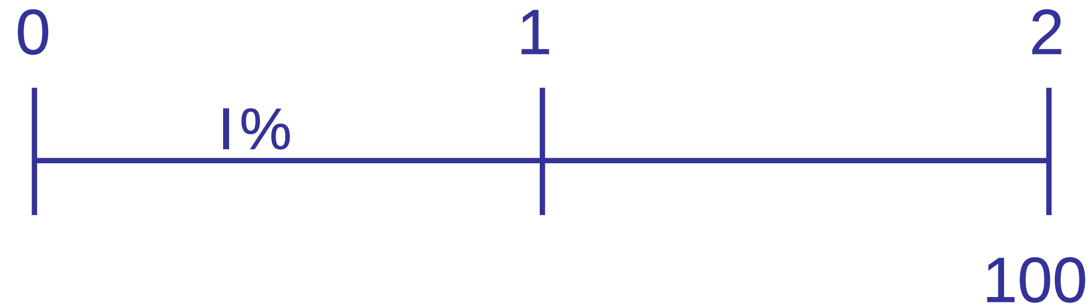
- Future value
- Present value
- Annuities
- Rates of return
- Amortization

Time lines

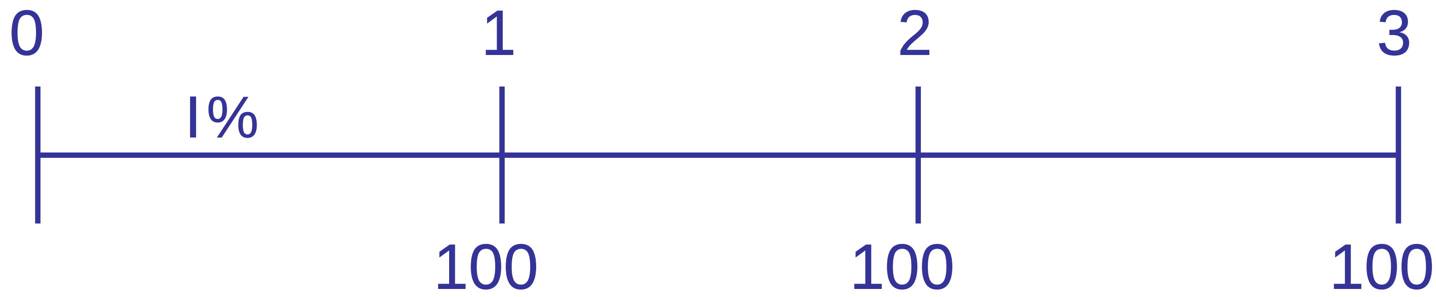
- Show the timing of cash flows.
- Tick marks occur at the end of periods, so Time 0 is today; Time 1 is the end of the first period (year, month, etc.) or the beginning of the second period.

Drawing time lines

$100 lump sum due in 2 years

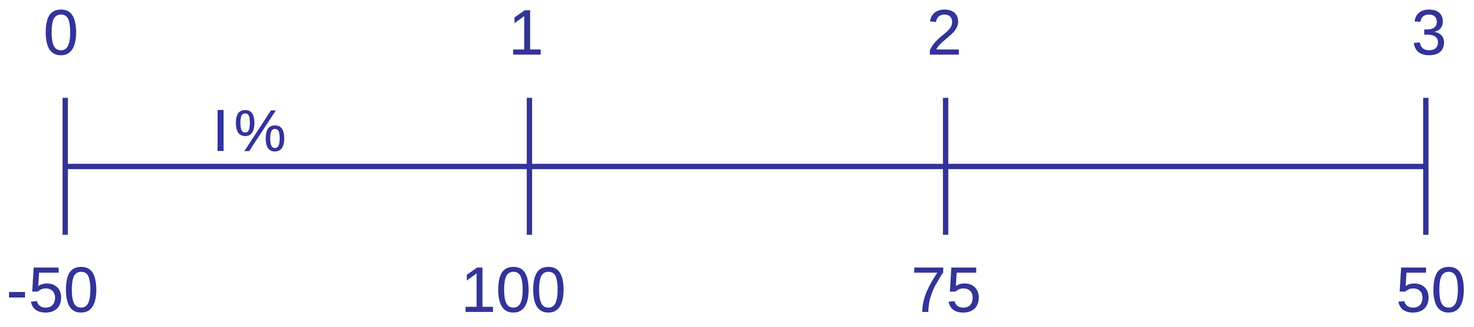


3 year $100 ordinary annuity



Drawing time lines

Uneven cash flow stream



What is the future value (FV) of an initial $100 after 3 years, if I/YR = 10%?

- Finding the FV of a cash flow or series of cash flows is called compounding.
- FV can be solved by using the step-by-step, financial calculator, and spreadsheet methods.

Solving for FV: The step-by-step and formula methods

- After 1 year:
  - $FV_1 = PV\ (1 + I) = \$100\ (1.10)$
    $= \$110.00$
- After 2 years:
  - $FV_2 = PV\ (1 + I)^2 = \$100\ (1.10)^2$
    $=\$121.00$
- After 3 years:
  - $FV_3 = PV\ (1 + I)^3 = \$100\ (1.10)^3$
    $=\$133.10$
- After N years (general case):
  - $FV_N = PV\ (1 + I)^N$

Solving for FV: The calculator method

- Solves the general FV equation.
- Requires 4 inputs into calculator, and will solve for the fifth. (Set to P/YR = 1 and END mode.)

INPUTS	**3**	**10**	**-100**	**0**	
	N	**I/YR**	**PV**	**PMT**	**FV**
OUTPUT					**133.10**

What is the present value (PV) of $100 due in 3 years, if I/YR = 10%?

- Finding the PV of a cash flow or series of cash flows is called discounting (the reverse of compounding).
- The PV shows the value of cash flows in terms of today’s purchasing power.

Solving for PV: The formula method

- Solve the general FV equation for PV:
  - $PV = FV_N / (1 + I)^N$
  - $PV = FV_3 / (1 + I)^3$
    $= \$100 / (1.10)^3$
    $= \$75.13$

Solving for PV: The calculator method

- Solves the general FV equation for PV.
- Exactly like solving for FV, except we have different input information and are solving for a different variable.

	N	I/YR	PV	PMT	FV
INPUTS	3	10		0	100
OUTPUT			-75.13		

Solving for I:
What interest rate would cause $100 to grow to $125.97 in 3 years?

- Solves the general FV equation for I.
- Hard to solve without a financial calculator or spreadsheet.

INPUTS	**3**		**-100**	**0**	**125.97**
	N	**I/YR**	**PV**	**PMT**	**FV**
OUTPUT		**8**			

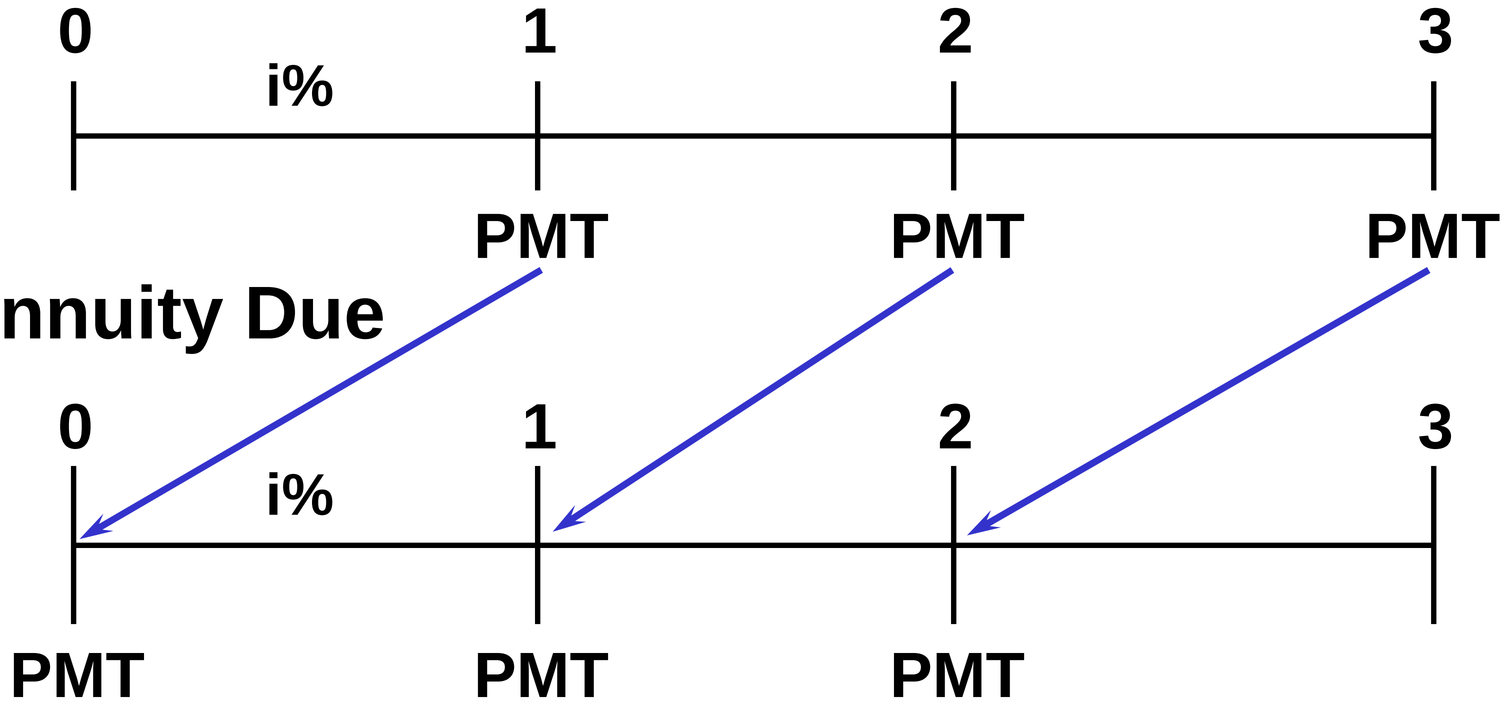
Solving for N:
If sales grow at 20% per year, how long before sales double?

- Solves the general FV equation for N.
- Hard to solve without a financial calculator or spreadsheet.

INPUTS		20	-1	0	2
	N	I/YR	PV	PMT	FV
OUTPUT	3.8				

What is the difference between an ordinary annuity and an annuity due?

Ordinary Annuity



Solving for FV:
3-year ordinary annuity of $100 at 10%

- $100 payments occur at the end of each period, but there is no PV.

INPUTS	3	10	0	-100	
	N	I/YR	PV	PMT	FV
OUTPUT					331

Solving for PV:
3-year ordinary annuity of $100 at 10%

- $100 payments still occur at the end of each period, but now there is no FV.

	N	I/YR	PV	PMT	FV
INPUTS	3	10		100	0
OUTPUT			-248.69		

Solving for FV:
3-year annuity due of $100 at 10%

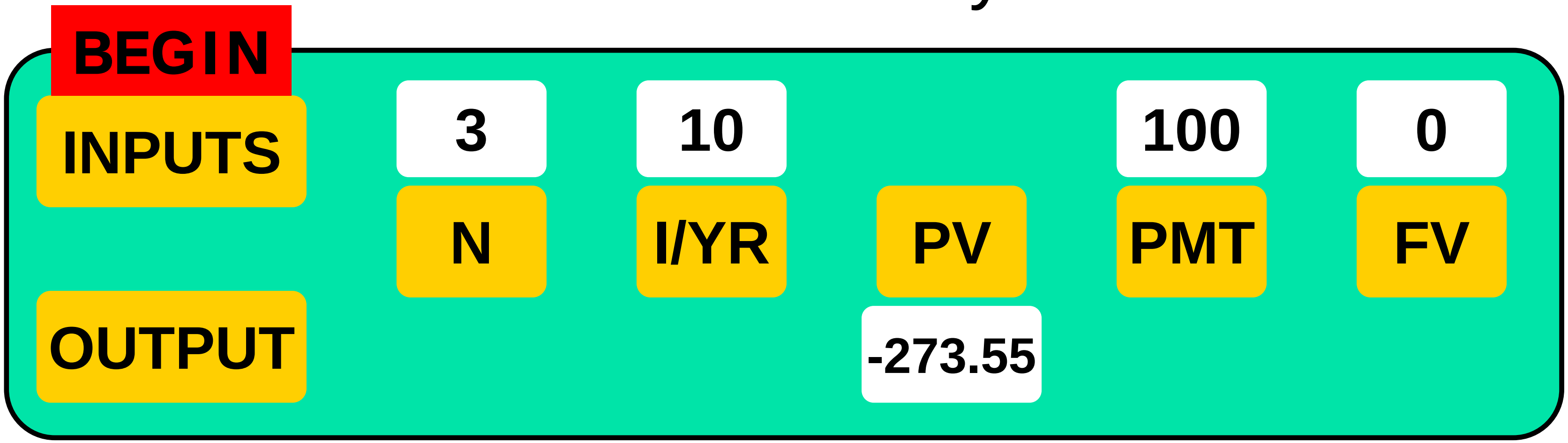
- Now, $100 payments occur at the beginning of each period.
- $FVA_{due} = FVA_{ord}(1+I) = \$331(1.10) = \$364.10.$
- Alternatively, set calculator to "BEGIN" mode and solve for the FV of the annuity:

BEGIN

INPUTS	3	10	0	-100	
	N	I/YR	PV	PMT	FV
OUTPUT					364.10

Solving for PV:
3-year annuity due of $100 at 10%

- Again, $100 payments occur at the beginning of each period.
- $PVA_{due} = PVA_{ord}(1+I) = \$248.69(1.10) = \$273.55.$
- Alternatively, set calculator to "BEGIN" mode and solve for the PV of the annuity:



What is the present value of a 5-year $100 ordinary annuity at 10%?

- Be sure your financial calculator is set back to END mode and solve for PV:
  - N = 5, I/YR = 10, PMT = 100, FV = 0.
  - PV = $379.08

What if it were a 10-year annuity? A 25-year annuity? A perpetuity?

- 10-year annuity
  - N = 10, I/YR = 10, PMT = 100, FV = 0; solve for PV = $614.46.
- 25-year annuity
  - N = 25, I/YR = 10, PMT = 100, FV = 0; solve for PV = $907.70.
- Perpetuity
  - PV = PMT / I = $100/0.1 = $1,000.

The Power of Compound Interest

A 20-year-old student wants to save $3 a day for her retirement. Every day she places $3 in a drawer. At the end of the year, she invests the accumulated savings ($1,095) in a brokerage account with an expected annual return of 12%.

How much money will she have when she is 65 years old?

Solving for FV: If she begins saving today, how much will she have when she is 65?

- If she sticks to her plan, she will have $1,487,261.89 when she is 65.

INPUTS	45	12	0	-1095	
	N	I/YR	PV	PMT	FV
OUTPUT					1,487,262

Solving for FV:
If you don't start saving until you are 40 years old, how much will you have at 65?

- If a 40-year-old investor begins saving today, and sticks to the plan, he or she will have $146,000.59 at age 65. This is $1.3 million less than if starting at age 20.
- Lesson: It pays to start saving early.

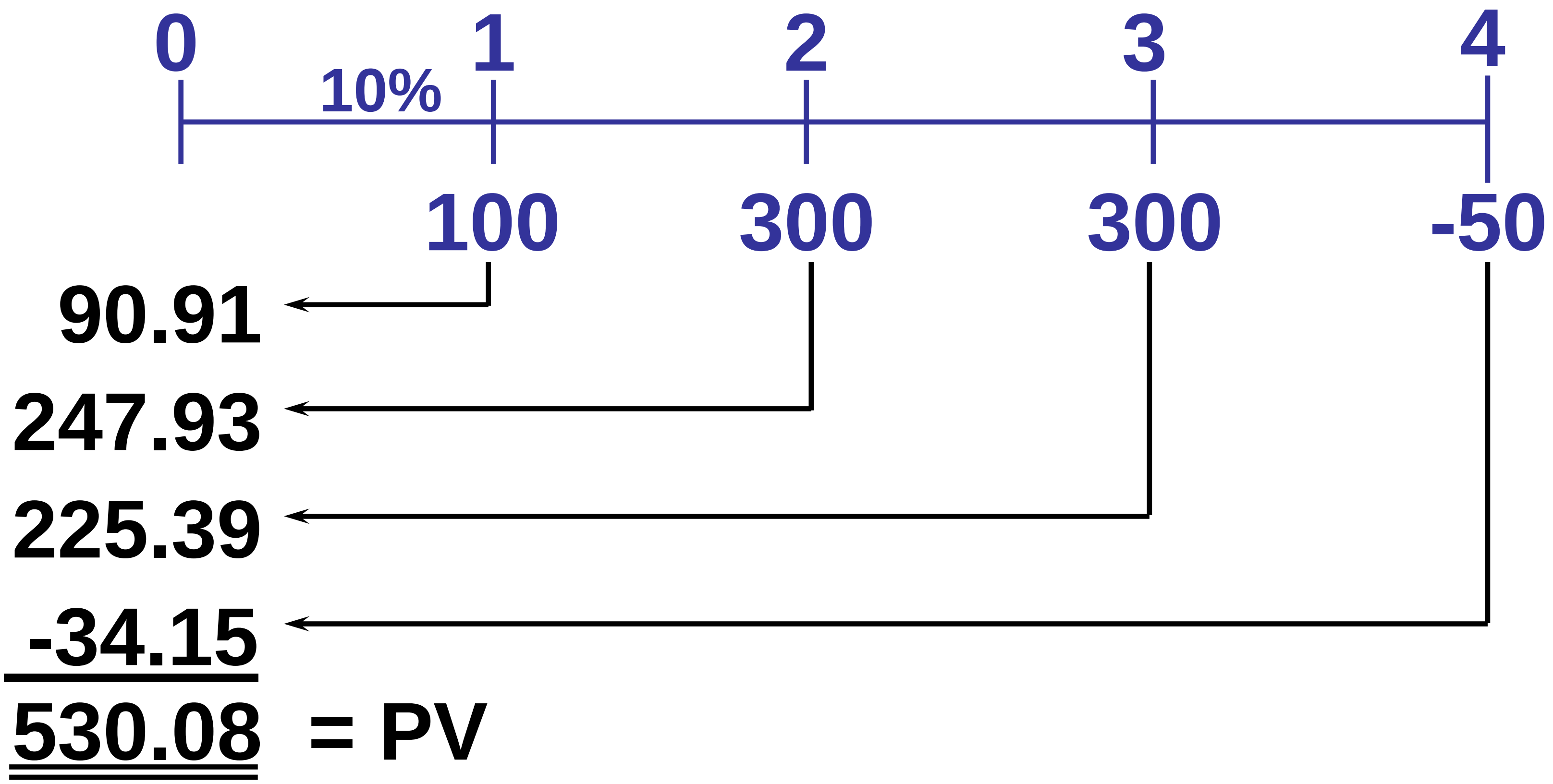
	N	I/YR	PV	PMT	FV
INPUTS	25	12	0	-1095	
OUTPUT					146,001

Solving for PMT: How much must the 40-year old deposit annually to catch the 20-year old?

- To find the required annual contribution, enter the number of years until retirement and the final goal of $1,487,261.89, and solve for PMT.

	N	I/YR	PV	PMT	FV
INPUTS	25	12	0		1,487,262
OUTPUT				-11,154.42	

What is the PV of this uneven cash flow stream?

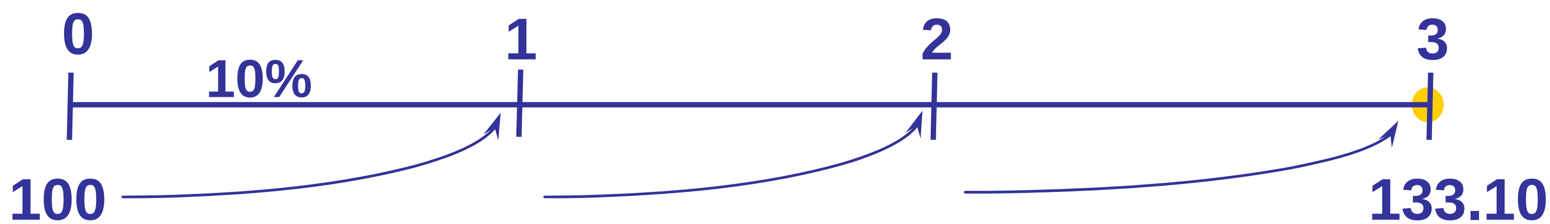


Solving for PV: Uneven cash flow stream

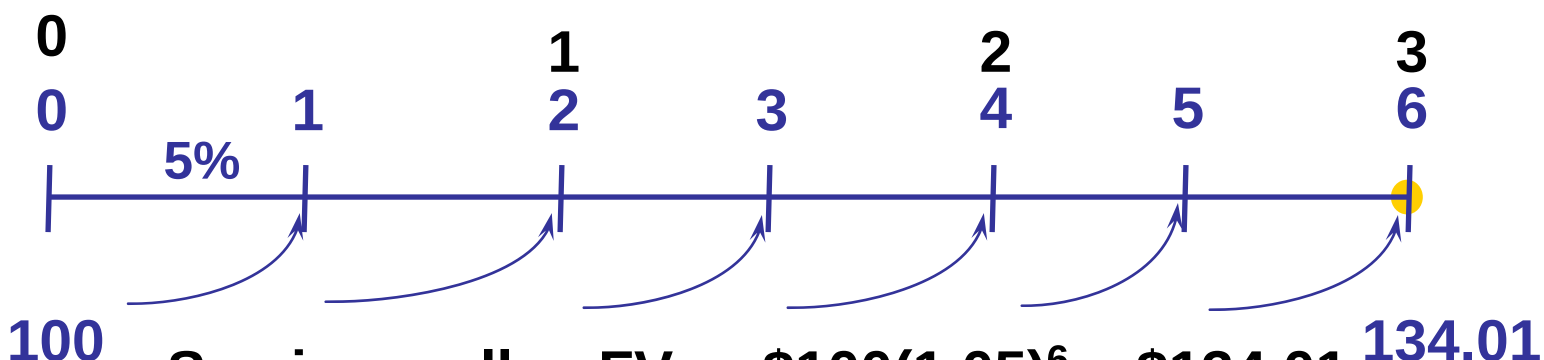
- Input cash flows in the calculator's "CFLO" register:
  - $CF_0 = 0$
  - $CF_1 = 100$
  - $CF_2 = 300$
  - $CF_3 = 300$
  - $CF_4 = -50$
- Enter I/YR = 10, press NPV button to get NPV = $530.087. (Here NPV = PV.)

Will the FV of a lump sum be larger or smaller if compounded more often, holding the stated I% constant?

- LARGER, as the more frequently compounding occurs, interest is earned on interest more often.



Annually: $FV_3 = \$100(1.10)^3 = \$133.10$



Semiannually: $FV_6 = \$100(1.05)^6 = \$134.01$

Classifications of interest rates

- Nominal rate (I_{NOM}) – also called the quoted or state rate. An annual rate that ignores compounding effects.
  - I_{NOM} is stated in contracts. Periods must also be given, e.g. 8% Quarterly or 8% Daily interest.
- Periodic rate (I_{PER}) – amount of interest charged each period, e.g. monthly or quarterly.
  - $I_{PER} = I_{NOM} / M$, where M is the number of compounding periods per year. M = 4 for quarterly and M = 12 for monthly compounding.

Classifications of interest rates

- Effective (or equivalent) annual rate (EAR = EFF%) – the annual rate of interest actually being earned, accounting for compounding.
  - EFF% for 10% semiannual investment

    $$\text{EFF\%} = (1 + I_{NOM} / M)^M - 1$$
    $$= (1 + 0.10 / 2)^2 - 1 = 10.25\%$$

  - Should be indifferent between receiving 10.25% annual interest and receiving 10% interest, compounded semiannually.

Why is it important to consider effective rates of return?

- Investments with different compounding intervals provide different effective returns.
- To compare investments with different compounding intervals, you must look at their effective returns (EFF% or EAR).
- See how the effective return varies between investments with the same nominal rate, but different compounding intervals.

EAR_{ANNUAL}	10.00%
$EAR_{QUARTERLY}$	10.38%
$EAR_{MONTHLY}$	10.47%
$EAR_{DAILY\ (365)}$	10.52%

When is each rate used?

- I_{NOM} written into contracts, quoted by banks and brokers. Not used in calculations or shown on time lines.
- I_{PER} Used in calculations and shown on time lines. If M = 1, $I_{NOM} = I_{PER} =$ EAR.
- EAR Used to compare returns on investments with different payments per year. Used in calculations when annuity payments don't match compounding periods.

What is the FV of $100 after 3 years under 10% semiannual compounding? Quarterly compounding?

$$FV_n = PV\left(1 + \frac{I_{NOM}}{M}\right)^{M \times N}$$

$$FV_{3S} = \$100\left(1 + \frac{0.10}{2}\right)^{2 \times 3}$$

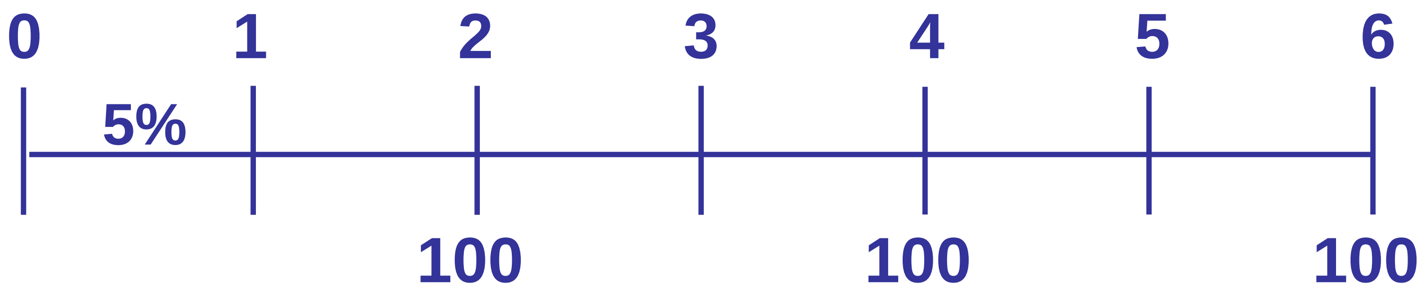
$$FV_{3S} = \$100(1.05)^6 = \$134.01$$

$$FV_{3Q} = \$100(1.025)^{12} = \$134.49$$

Can the effective rate ever be equal to the nominal rate?

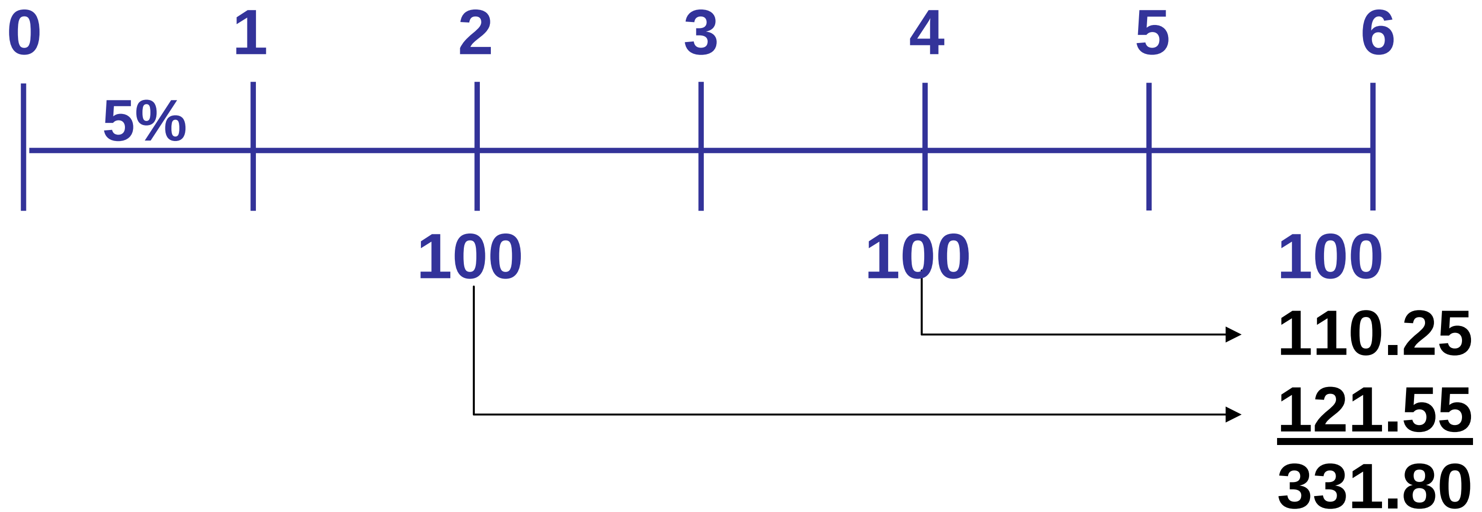
- Yes, but only if annual compounding is used, i.e., if M = 1.
- If M > 1, EFF% will always be greater than the nominal rate.

What's the FV of a 3-year $100 annuity, if the quoted interest rate is 10%, compounded semiannually?



- Payments occur annually, but compounding occurs every 6 months.
- Cannot use normal annuity valuation techniques.

Method 1: Compound each cash flow



$$FV_3 = \$100(1.05)^4 + \$100(1.05)^2 + \$100$$

$$FV_3 = \$331.80$$

Method 2: Financial calculator

- Find the EAR and treat as an annuity.
- EAR = $(1 + 0.10 / 2)^2 – 1 = 10.25\%$.

INPUTS	3	10.25	0	-100	
	N	I/YR	PV	PMT	FV
OUTPUT					331.80

Find the PV of this 3-year ordinary annuity.

- Could solve by discounting each cash flow, or ...
- Use the EAR and treat as an annuity to solve for PV.

	N	I/YR	PV	PMT	FV
INPUTS	3	10.25		100	0
OUTPUT			-247.59		

Loan amortization

- Amortization tables are widely used for home mortgages, auto loans, business loans, retirement plans, etc.
- Financial calculators and spreadsheets are great for setting up amortization tables.
- EXAMPLE: Construct an amortization schedule for a $1,000, 10% annual rate loan with 3 equal payments.

Step 1:
Find the required annual payment

- All input information is already given, just remember that the FV = 0 because the reason for amortizing the loan and making payments is to retire the loan.

INPUTS	3	10	-1000		0
	N	I/YR	PV	PMT	FV
OUTPUT				402.11	

Step 2:
Find the interest paid in Year 1

- The borrower will owe interest upon the initial balance at the end of the first year. Interest to be paid in the first year can be found by multiplying the beginning balance by the interest rate.

$$INT_t = \text{Beg bal}_t \, (I)$$

$$INT_1 = \$1{,}000 \, (0.10) = \$100$$

Step 3:
Find the principal repaid in Year 1

- If a payment of $402.11 was made at the end of the first year and $100 was paid toward interest, the remaining value must represent the amount of principal repaid.

$$PRIN = PMT – INT$$

$$= \$402.11 - \$100 = \$302.11$$

Step 4:
Find the ending balance after Year 1

- To find the balance at the end of the period, subtract the amount paid toward principal from the beginning balance.

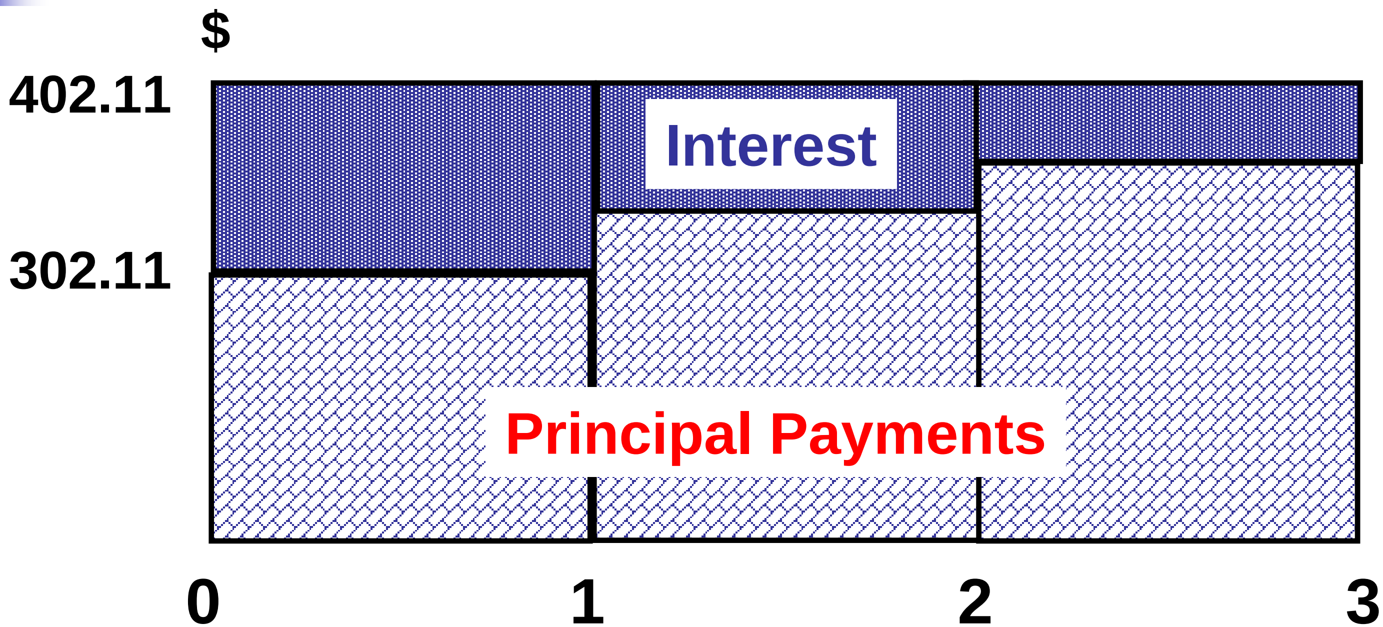
$$
\begin{aligned}
\text{END BAL} &= \text{BEG BAL} - \text{PRIN} \\
&= \$1{,}000 - \$302.11 \\
&= \$697.89
\end{aligned}
$$

Constructing an amortization table: Repeat steps 1 – 4 until end of loan

Year	BEG BAL	PMT	INT	PRIN	END BAL
1	$1,000	$402	$100	$302	$698
2	698	402	70	332	366
3	366	402	37	366	0
TOTAL		1,206.34	206.34	1,000	-

- Interest paid declines with each payment as the balance declines. What are the tax implications of this?

Illustrating an amortized payment: Where does the money go?



- Constant payments.
- Declining interest payments.
- Declining balance.